THE ANALYSIS OF CONTINUOUS RECTANGULAR PLATES BY CARRY-OVER MOMENTS

By

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CARRY-OVER MOMENTS

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PREFACE

The material presented in this thesis is the outgrowth of lectures delivered by Dr. Kerry S. Havner in the fall of 1960 and the many consultations with both him and Professor Jan J. Tuma.

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NOMENCLATURE

a	Length of Plate.
a_j, a_k,	Lengths of Spans J and K, respectively.
b	Width of Plate.
f_{xn}, f_{ijn}, f_{jin}	Angular Flexibilities for the nth Component.
g_{xn}, g_{ijn}, g_{jin}	Angular Carry-Over Values for nth Component.
h	Thickness of Plate.
n	Term in an Infinite Trigonometric Series.
$p(x, y)$	Load Distribution.
r_{ijn}, r_{kjn}	Carry-Over Factors for nth Component.
t_{xn}, t_{ijn}, t_{jin}	Angular Load Functions for nth Component.
w	Intensity of Load.
x^*_{jn}	Amplitude of Starting Moment at j for the nth Component.
x, y	Coordinate Axes.
A_n, B_n, C_n, D_n	Coefficients of Hyperbolic Functions.
D	Flexural Rigidity of Plate.
C_1	Angular Flexibility Coefficient.
C_2	Angular Carry-Over Value Coefficient.
C_3, C_4, C_5	End Slope Coefficients.
H	Homogeneous Solution.

J, K . . . . Spans of Continuous Plate.

M, $M_i(y)$, $M_j(y)$, $M_k(y)$. Moments in x-Direction as Functions of y.

P . . . . Particular Solution.

$P_n(x)$, P_n. . . . . Load Functions for nth Component.

P_c. . . . . Concentrated Load.

W_j, W_k, W . . . . Deflection Surfaces.

W_o. . . . . Deflection Surface Due to Loads.

γ, λ . . . . Proportions of Width and Length, Respectively, of Plate.

μ . . . . Poisson's Ratio.

ϕ_{ijn}, ϕ_{jin} . . . . Amplitudes of Slopes at i and j for nth Component.

ϕ_n . . . . Deflection Function for nth Component Due to Loads.

ψ_n . . . . Deflection Function for nth Component Due to Moments.

$$\beta_n = \frac{n\pi}{b}.$$

CHAPTER I

INTRODUCTION

The purpose of this thesis is to show the development of the angular functions for thin rectangular plates of constant rigidity and to extend the applicability of the Carry-Over Moment Method of analysis to one-way continuous rectangular plates.

The method developed in this thesis is applicable to rectangular plates with rigid simple supports on two opposite edges. The other two edges may be either free, fixed, simply-supported, or continuous over a rigid simple support. The plate may be continuous over any number and spacing of intermediate rigid simple supports which lie transverse to the simply-supported edges. The flexural rigidity in any span must be constant throughout that span and the assumptions of simple bending theory of plates are made. It is also assumed that there exists no resultant horizontal direct stress on any vertical cross-section of the plate. The sign convention of deformation is adopted.

The essential features of the carry-over procedure presented in this thesis are similiar to, and derived from, the Carry-Over Moment Method of analysis developed by Tuma (2). The numerical coefficients tabulated in Chapter IV are used to calculate the angular functions, carry-over functions, and starting values for the carry-over procedures.

Continuous plates have been analyzed by many different methods. The disadvantage of most of these methods is the time required to obtain solutions. Newmark (8) extended the Moment-Distribution Method to plates continuous in one direction over flexible supports. The three-moment equations have been used by Girkmann (1) and Timoshenko (7) to solve certain continuous rectangular plates.

CHAPTER II

ANGULAR FUNCTIONS

2-1 Angular Functions Due to End Moments

Consider a simply-supported rectangular plate with lengths a and b in the directions x and y, respectively, to be acted upon by a moment that varies as a function of y along the edge $x = 0$. The equilibrium configuration of the plate may be analyzed by expressing the moment variation as a trigonometric series of the form

$$M_{x=0} = X_1 \sin \frac{\pi y}{b} + X_2 \sin \frac{2\pi y}{b} + \cdots + X_n \sin \frac{n\pi y}{b} + \cdots$$

$$= \sum_{n=1}^{\infty} X_n \sin \beta_n y, \quad \text{where } \beta_n = \frac{n\pi}{b}, \tag{2-1}$$

and superimposing the results obtained from each component of the variation (Fig. 2-1).

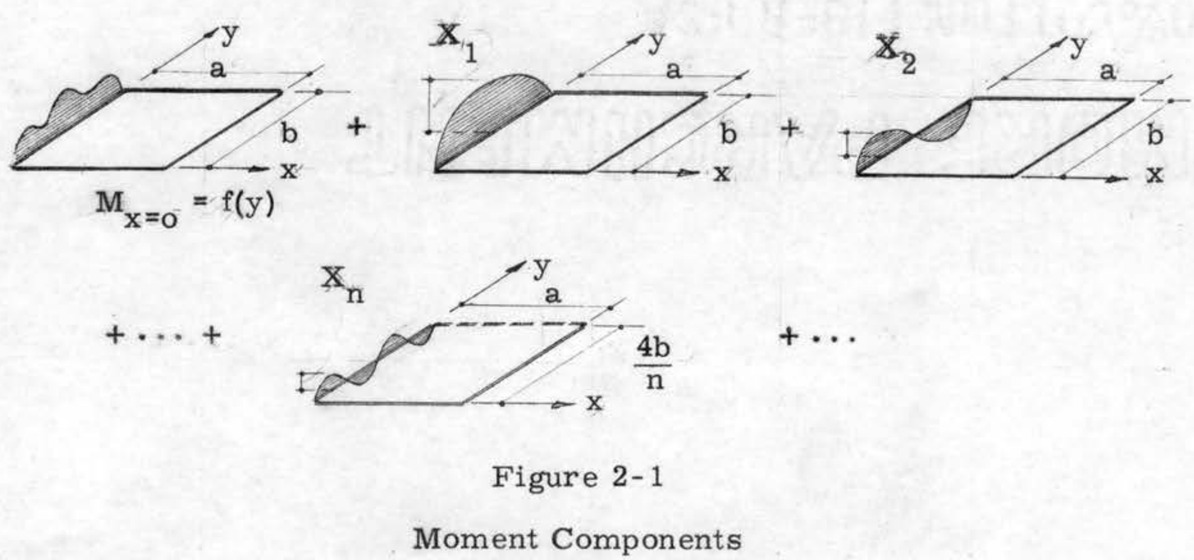


Figure 2-1

Moment Components

Choosing the representative component to be the nth term, the moment along the edge $x = 0$ becomes

$$(M_{x=0})_n = X_n \sin \beta_n y. \tag{2-2}$$

The deflection surface $W_n(x, y)$ produced by this moment is the function which satisfies the biharmonic differential equation

$$\nabla^4 W_n = \frac{\partial^4 W_n}{\partial x^4} + 2 \frac{\partial^4 W_n}{\partial x^2 \partial y^2} + \frac{\partial^4 W_n}{\partial y^4} = 0 \tag{2-3}$$

and the boundary conditions

$$\left.\begin{aligned}
x = 0 ;\quad W_n &= 0 ,\quad -D\left(\frac{\partial^2 W_n}{\partial x^2} + \mu \frac{\partial^2 W_n}{\partial y^2}\right) = \\
&= X_n \sin \beta_n y \\
x = a ;\quad W_n &= 0,\quad -D\left(\frac{\partial^2 W_n}{\partial x^2} + \mu \frac{\partial^2 W_n}{\partial y^2}\right) = 0 \\
y = 0, b ;\quad W_n &= 0,\quad -D\left(\frac{\partial^2 W_n}{\partial y^2} + \mu \frac{\partial^2 W_n}{\partial x^2}\right) = 0.
\end{aligned}\right\} \tag{2-4a}$$

Noting that along the edges $x = 0$ and $x = a$ the curvature in the y- direction, $\frac{\partial^2 W_n}{\partial y^2}$, is zero and that along the edges $y = 0$ and $y = b$ the curvature in the x- direction, $\frac{\partial^2 W_n}{\partial x^2}$, is zero, the boundary conditions reduce to

$$\left.\begin{aligned}
x = 0;\quad W_n &= 0,\quad \frac{\partial^2 W_n}{\partial x^2} = -\frac{X_n}{D} \sin \beta_n y \\
x = a;\quad W_n &= 0,\quad \frac{\partial^2 W_n}{\partial x^2} = 0
\end{aligned}\right\} \tag{2-4b}$$

$$y = 0, b; \quad W_n = 0, \quad \frac{\partial^2 W_n}{\partial y^2} = 0.$$

Assuming a deflection surface of the form

$$W_n = \psi_n(x) \sin \beta_n y \tag{2-5}$$

where $\psi_n(x)$ is independent of y, the partial derivatives are

$$\frac{\partial^4 W_n}{\partial x} = \frac{d^4 \psi_n(x)}{dx^4} \sin \beta_n y$$

$$\frac{\partial^4 W_n}{\partial x^2 \partial y^2} = - \beta_n^2 \frac{d^2 \psi_n(x)}{dx^2} \sin \beta_n y$$

$$\frac{\partial^4 W_n}{\partial y^4} = \beta_n^4 \psi_n(x) \sin \beta_n y.$$

Substitution of these partial derivatives into Equation (2-3) yields

$$\left\{ \frac{d^4 \psi_n(x)}{dx^4} - 2 \beta_n^2 \frac{d^2 \psi_n(x)}{dx^2} + \beta_n^4 \psi_n(x) \right\} \sin \beta_n y = 0 .$$

Since the above expression must be identically satisfied for all y in the range $0 \leq y \leq b$, the governing differential equation for $\psi_n(x)$ becomes

$$\frac{d^4 \psi_n(x)}{dx^4} - 2\beta_n^2 \frac{d^2 \psi_n(x)}{dx^2} + \beta_n^4 \psi_n(x) = 0. \tag{2-6}$$

The general solution of Equation (2-6) may be written in the form

$$\psi_n(x) = A_n \cosh \beta_n x + \beta_n x B_n \sinh \beta_n x +$$

$$+ C_n \sinh \beta_n x + \beta_n x D_n \cosh \beta_n x \tag{2-7}$$

where the constants A_n, B_n, C_n, and D_n are to be evaluated from the first four boundary conditions, Equations (2-4b).

Substituting Equation (2-7) into Equation (2-5), the expression for the deflection surface becomes

$$W_n = \left[A_n \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x + C_n \sinh \beta_n x + \right.$$
$$\left. + \beta_n x\, D_n \cosh \beta_n x\right] \sin \beta_n y. \qquad (2\text{-}8)$$

The first two partial derivatives of Equation (2-8) with respect to x are:

$$\frac{\partial W_n}{\partial x} = \beta_n \left[(A_n + B_n) \sinh \beta_n x + \beta_n x\, B_n \cosh \beta_n x + \right.$$
$$\left. + (C_n + D_n) \cosh \beta_n x + \beta_n x\, D_n \sinh \beta_n x\right] \sin \beta_n y$$

$$\frac{\partial^2 W_n}{\partial x^2} = \beta_n^2 \left[(A_n + 2B_n) \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x \right.$$
$$\left. + (C_n + 2D_n) \sinh \beta_n x + \beta_n x\, D_n \cosh \beta_n x\right] \sin \beta_n y.$$

From Equations (2-4b),

$\underline{x = 0}$;

$$W_n = 0 = \left[A_n \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x + \right.$$
$$\left. + C_n \sinh \beta_n x + \beta_n x\, D_n \cosh \beta_n x\right]_{x=0} \sin \beta_n y$$

$$0 = A_n,$$

$$\frac{\partial^2 W_n}{\partial x^2} = -\frac{X_n}{D} \sin \beta_n y = \beta_n^2 \left[(\cancel{A_n} + 2B_n) \cosh \beta_n x + \right.$$
$$+ \beta_n x\, B_n \sinh \beta_n x + (C_n + 2D_n) \sinh \beta_n x +$$
$$\left. + \beta_n x\, D_n \cosh \beta_n x\right]_{x=0} \sin \beta_n y$$

$$-\frac{X_n}{D} = \beta_n^2 (\cancel{A_n} + 2B_n) = 2B_n \beta_n^2 \longrightarrow B_n = -\frac{X_n}{2D\beta_n^2},$$

<u>x = a;</u>

$$W_n = 0 = \Big[\cancel{A_n} \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x + C_n \sinh \beta_n x +$$

$$+ \beta_n x\, D_n \cosh \beta_n x\Big]_{x=a} \sin \beta_n y$$

$$0 = \beta_n a\, B_n \sinh \beta_n a + C_n \sinh \beta_n a +$$

$$+ \beta_n a\, D_n \cosh \beta_n a$$

$$\frac{\partial^2 W_n}{\partial x^2} = 0 = \beta_n^2 \Big[(\cancel{A_n} + 2B_n) \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x +$$

$$+ (C_n + 2D_n) \sinh \beta_n x +$$

$$+ \beta_n x\, D_n \cosh \beta_n x\Big]_{x=a} \sin \beta_n y$$

$$0 = 2B_n \cosh \beta_n a + \beta_n a\, B_n \sinh \beta_n a +$$

$$+ (C_n = 2D_n) \sinh \beta_n a + \beta_n a\, D_n \cosh \beta_n a$$

$$0 = (\beta_n a\, B_n \sinh \beta_n a + C_n \sinh \beta_n a +$$

$$+ \beta_n a\, D_n \cosh \beta_n a) + 2B_n \cosh \beta_n a +$$

$$+ 2D_n \sinh \beta_n a$$

$$0 = 0 + 2B_n \cosh \beta_n a + 2D_n \sinh \beta_n a$$

$$0 = -\frac{X_n}{2D\beta_n^2} \coth \beta_n a + D_n \Longrightarrow$$

$$D_n = \frac{X_n}{2D\beta_n^2} \coth \beta_n a.$$

From above

$$0 = \beta_n a\, B_n \sinh \beta_n a + C_n \sinh \beta_n a +$$

$$+ \beta_n a\, D_n \cosh \beta_n a$$

$$0 = \beta_n a\,(B_n + D_n \coth \beta_n a) + C_n$$

$$0 = \beta_n a\, B_n\,(1 - \coth^2 \beta_n a) + C_n$$

$$0 = \beta_n a \left(-\frac{X_n}{2D\,\beta_n^2}\right)\left(\frac{-1}{\sinh^2 \beta_n a}\right) + C_n$$

$$\longrightarrow C_n = \left(-\frac{aX_n}{2D\beta_n}\right)\frac{1}{\sinh^2 \beta_n a} \;.$$

The expressions for the constants A_n, B_n, C_n, and D_n have thus been found to be

$$\left.\begin{array}{ll} A_n = 0 & C_n = -\dfrac{aX_n}{2D\,\beta_n}\,\dfrac{1}{\sinh^2 \beta_n a} \\ B_n = -\dfrac{X_n}{2D\,\beta_n^2} & D_n = \dfrac{X_n}{2D\,\beta_n^2}\coth \beta_n a \;. \end{array}\right\} \quad (2\text{-}9)$$

Substitution of Equations (2-9) into Equation (2-8) yields

$$W_n = \frac{X_n}{2D\,\beta_n^2 \sinh \beta_n a}\left\{\beta_n x(\cosh \beta_n a \cosh \beta_n x - \right.$$

$$\left. - \sinh \beta_n a \sinh \beta_n x) - \beta_n a\,\frac{\sinh \beta_n x}{\sinh \beta_n a}\right\} \sin \beta_n y$$

$$W_n = \frac{X_n}{2D\,\beta_n^2 \sinh \beta_n a}\left\{\beta_n x \cosh \beta_n (a - x) - \right.$$

$$\left. - \beta_n a\,\frac{\sinh \beta_n x}{\sinh \beta_n a}\right\} \sin \beta_n y. \qquad (2\text{-}10)$$

The end slopes at the edges $x = 0$ and $x = a$ may now be determined. The first partial derivative of the deflection surface, W_n, with respect to x is

$$\frac{\partial W_n}{\partial x} = \frac{X_n}{2D\beta_n \sinh\beta_n a}\left\{\cosh\beta_n(a-x) - \right.$$

$$\left. - \beta_n x \sinh\beta_n(a-x) - \beta_n a \frac{\cosh\beta_n x}{\sinh\beta_n a}\right\} \sin\beta_n y. \tag{2-11}$$

The slopes at $x = 0$ and $x = a$ are:

$$\left(\frac{\partial W_n}{\partial x}\right)_{x=0} = X_n\left\{\frac{1}{2D\beta_n}\left(\coth\beta_n a - \frac{\beta_n a}{\sinh^2\beta_n a}\right)\right\} \sin\beta_n y \tag{2-12}$$

$$-\left(\frac{\partial W_n}{\partial x}\right)_{x=a} = X_n\left\{\frac{\beta_n a \coth\beta_n a - 1}{2D\beta_n \sinh\beta_n a}\right\} \sin\beta_n y\ . \tag{2-13}$$

Denoting

$$f_{xn} = \frac{1}{2D\beta_n}\left\{\coth\beta_n a - \frac{\beta_n a}{\sinh^2\beta_n a}\right\}$$

$$g_{xn} = \frac{\beta_n a \coth\beta_n a - 1}{2D\beta_n \sinh\beta_n a}\ , \tag{2-14}$$

the slopes become

$$\left(\frac{\partial W_n}{\partial x}\right)_{x=0} = X_n f_{xn} \sin\beta_n y \tag{2-15}$$

$$-\left(\frac{\partial W_n}{\partial x}\right)_{x=a} = X_n g_{xn} \sin\beta_n y\ . \tag{2-16}$$

The quantities f_{xn} and g_{xn} are thus influence coefficients for the slopes at $x = 0$ and $x = a$, respectively, due to the nth mode of moments acting at the edge $x = 0$ and are defined to be the nth-component angular functions of the plate in the x-direction due to end moments. The physical interpretation of these quantities follows.

f_{xn} is the maximum slope per unit length at the edge of a simply-supported plate due to a unit-amplitude sinusoidal moment at that edge. f_{xn} is called the nth component angular flexibility.

g_{xn} is the maximum slope per unit length at the edge of a simply-supported plate due to a unit-amplitude sinusoidal moment at the far edge. g_{xn} is called the nth component angular carry-over value.

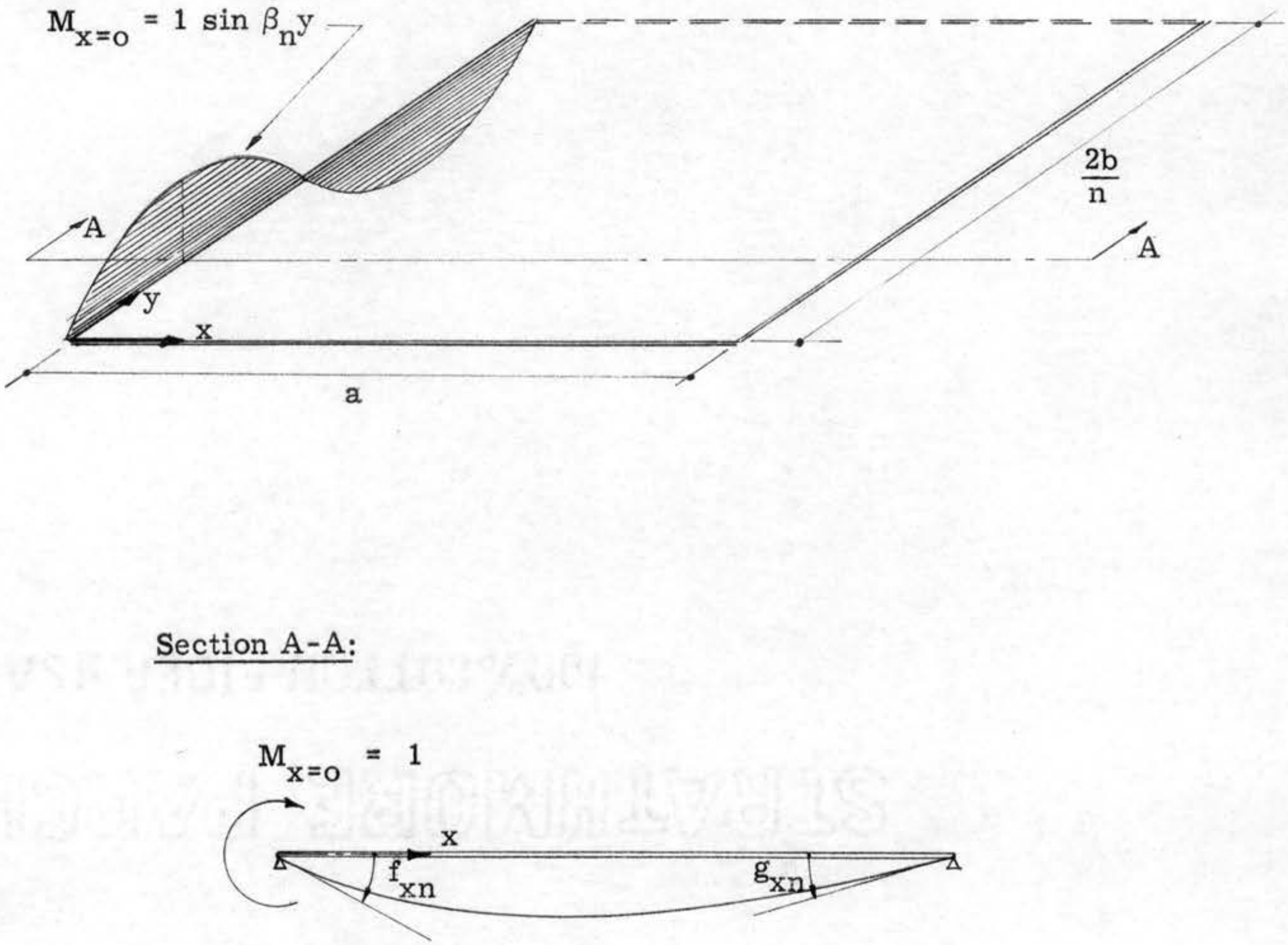


Figure 2-2

The nth-Component Angular Functions Due to End Moments

2-2 Angular Load Functions

In general, any load distribution on a rectangular plate can be expressed as a trigonometric series of the form

$$p(x, y) = \sum_{n=1}^{\infty} P_n(x) \sin \beta_n y , \quad \beta_n = \frac{n\pi}{b} \qquad (2\text{-}17)$$

where $P_n(x)$ is a function of x and may, in fact, be a trigonometric series itself.

The equilibrium configuration of the plate produced by this load distribution may be analyzed in component parts corresponding to the various load components of the series (Equation 2-17) and superimposing these parts (Fig. 2-3).

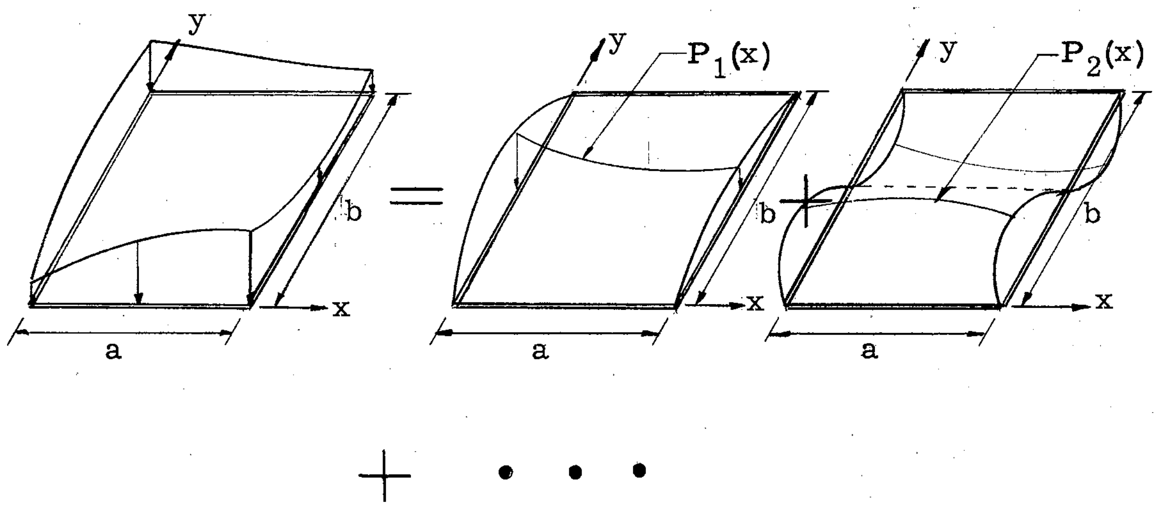


Figure 2-3

Load Components

Angular load functions for two types of load distributions are to be considered:

1.) A one-directional load variation in the y-direction.

2.) A concentrated load at any point (x, y) on the plate.

One-Directional Load Variation. For a one-directional load variation in the y-direction, the load is expressible in the form

$$p(x, y) = \sum_{n=1}^{\infty} P_n \sin \beta_n y \tag{2-18}$$

where P_n is independent of x and y. The constants P_n are the Fourier constants for the load representation and have values

$$P_n = \frac{2}{b} \int_0^b p(x, y) \sin \beta_n y \, dy . \tag{2-19}$$

It is convenient, for this case of loading, to choose a set of corrdinate axes as shown in fig. 2-4.

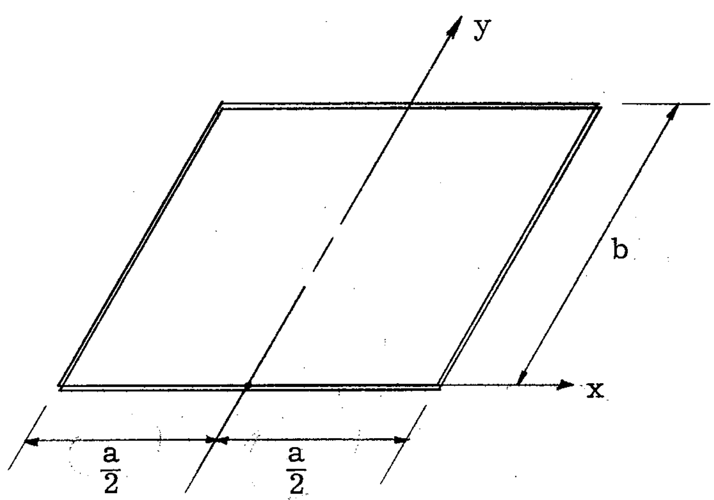


Figure 2-4. Coordinate Axes for Solution with One-Directional Load Variation.

For simple supports on all four edges of the plate of Fig. 2-4 and a representative nth component of loading, the deflection surface $W_{on}(x, y)$ produced must satisfy the differential equation

$$\nabla^4 W_{on} = \frac{\partial^4 W_{on}}{\partial x^4} + 2\frac{\partial^4 W_{on}}{\partial x^2 \partial y^2} + \frac{\partial^4 W_{on}}{\partial y^4} =$$

$$= \frac{p_n(x, y)}{D} = \frac{P_n \sin \beta_n y}{D} \tag{2-20}$$

and the boundary conditions

$$x = \pm a/2 ; \quad W_{on} = 0, \quad \frac{\partial^2 W_{on}}{\partial x^2} = 0$$

$$y = 0, b; \quad W_{on} = 0, \quad \frac{\partial^2 W_{on}}{\partial y^2} = 0 . \tag{2-21}$$

Assuming the deflection surface to be of the form

$$W_{on} = \phi_n(x) \sin \beta_n y \tag{2-22}$$

where $\phi_n(x)$ is independent of y, the partial derivatives are

$$\frac{\partial^4 W_{on}}{\partial x^4} = \frac{d^4 \phi_n(x)}{dx^4} \sin \beta_n y$$

$$\frac{\partial^4 W_{on}}{\partial x^2 \partial y^2} = -\beta_n^2 \frac{d^2 \phi_n(x)}{dx^2} \sin \beta_n y$$

$$\frac{\partial^4 W_{on}}{\partial y^4} = \beta_n^4 \phi_n(x) \sin \beta_n y .$$

Substitution of these partial derivatives into Equation (2-19) yields the governing differential equation for $\phi_n(x)$,

$$\frac{d^4 \phi_n(x)}{dx^4} - 2\beta_n^2 \frac{d^2 \phi_n(x)}{dx^2} + \beta_n^4 \phi_n(x) = \frac{P_n}{D} . \tag{2-23}$$

The general solution of Equation (2-22) is composed of a homogeneous plus a particular solution. The homogeneous solution is given by

$$(\phi_n(x))_H = A_n \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x + C_n \sinh \beta_n x + \beta_n x\, D_n \cosh \beta_n x, \qquad (2\text{-}24)$$

and the particular solution is taken to be the wide beam solution

$$(\phi_n(x))_P = \frac{P_n}{D\beta_n^4} . \qquad (2\text{-}25)$$

Thus

$$\phi_n(x) = A_n \cosh \beta_n x + \beta_n B_n \sinh \beta_n x + C_n \sinh \beta_n x + \beta_n x\, D_n \cosh \beta_n x + \frac{P_n}{D\beta_n^4} . \qquad (2\text{-}26)$$

Noting that the load variation, geometry of the plate, and boundary conditions are all symmetrical with respect to the y-axis, then the deflection surface, W_{on}, and thus $\phi_n(x)$ must also be symmetrical with respect to the y-axis. This implies that

$$C_n = 0, \qquad D_n = 0 .$$

Thus, after substituting Equation (2-26) into Equation (2-22), the deflection surface becomes

$$W_{on} = \left[A_n \cosh \beta_n x + \beta_n x\, B_n \sinh \beta_n x + \frac{P_n}{D\beta_n^4}\right] \sin \beta_n y. \qquad (2\text{-}27)$$

From the boundary conditions, Equations (2-20),

$$x = \frac{a}{2} ;$$

$$W_{on} = 0 = \left[A_n \cosh \frac{\beta_n a}{2} + \frac{\beta_n a}{2} B_n \sinh \frac{\beta_n a}{2} + \frac{P_n}{D\beta_n^4}\right] \sin \beta_n y .$$

$$\frac{\partial^2 W_{on}}{\partial x^2} = 0 = \beta_n^2 \left[(A_n + 2B_n) \cosh \frac{\beta_n a}{2} + \right.$$

$$\left. + \frac{\beta_n a}{2} B_n \sinh \frac{\beta_n a}{2} \right] \sin \beta_n y .$$

It follows that

$$A_n \left(\cosh \frac{\beta_n a}{2} \right) + B_n \left(\frac{\beta_n a}{2} \sinh \frac{\beta_n a}{2} \right) +$$

$$+ \frac{P_n}{D\beta_n^4} = 0$$

and

$$A_n \left(\cosh \frac{\beta_n a}{2} \right) + B_n \left(2 \cosh \frac{\beta_n a}{2} + \frac{\beta_n a}{2} \sinh \frac{\beta_n a}{2} \right) = 0 ,$$

from which

$$\left.\begin{aligned} B_n &= \frac{P_n}{2D\beta_n^4 \cosh \frac{\beta_n a}{2}} \\ A_n &= \frac{-P_n \left(2 - \frac{\beta_n a}{2} \tanh \frac{\beta_n a}{2} \right)}{2D\beta_n^4 \cosh \frac{\beta_n a}{2}} . \end{aligned}\right\} \quad (2\text{-}28)$$

With these being the expressions for the constants A_n and B_n, the final form of the deflection surface is

$$W_{on} = \frac{P_n}{D\beta_n^4} \left\{ \frac{1}{2 \cosh \frac{\beta_n a}{2}} \left[\beta_n x \sinh \beta_n x - \right. \right.$$

$$\left. \left. - \left(2 + \frac{\beta_n a}{2} \tanh \frac{\beta_n a}{2} \right) \cosh \beta_n x \right] + 1 \right\} \sin \beta_n y . \quad (2\text{-}29)$$

The first partial derivative of the deflection surface with respect to x is

$$\frac{\partial W_{on}}{\partial x} = \frac{P_n}{D\beta_n^3}\left\{\frac{1}{2\cosh\frac{\beta_n a}{2}}\left[\beta_n x \cosh \beta_n x - \left(1 + \frac{\beta_n a}{2}\tanh\frac{\beta_n a}{2}\right)\sinh \beta_n x\right]\right\}\sin \beta_n y \;. \tag{2-30}$$

Evaluating Equation (2-31) at $x = -\frac{a}{2}$ gives the following variation in slope along that edge:

$$\left(\frac{\partial W_{on}}{\partial x}\right)_{x=-\frac{a}{2}} = \frac{P_n}{2D\beta_n^3}\left[\left(1 + \frac{\beta_n a}{2}\tanh\frac{\beta_n a}{2}\right)\tanh\frac{\beta_n a}{2} - \frac{\beta_n a}{2}\right]\sin \beta_n y \;. \tag{2-31}$$

Denoting

$$t_{xn} = \frac{P_n}{2D\beta_n^3}\left[\left(1 + \frac{\beta_n a}{2}\tanh\frac{\beta_n a}{2}\right)\tanh\frac{\beta_n a}{2} - \frac{\beta_n a}{2}\right] , \tag{2-32}$$

Equation (2-32) becomes

$$\left(\frac{\partial W_{on}}{\partial x}\right)_{x=-\frac{a}{2}} = t_{xn} \sin \beta_n y \;. \tag{2-33}$$

Thus for the nth component of the load variation with an amplitude of P_n the corresponding amplitude of the slope variation at the edge $x = -\frac{a}{2}$ becomes t_{xn}. The quantity t_{xn} is therefore defined to be an nth-component angular load function of the plate in the x-direction.

Concentrated Load at the Point $x = \lambda a$, $y = \gamma b$. The formulation of the angular load functions of a plate due to a concentrated load at the point $x = \lambda a$ and $y = \gamma b$ may be accomplished in the following manner:

Consider a simply-supported plate to be acted upon by a sinusoidal moment variation along the edge $x = 0$ and expressed by the equation

$$M_{x=0} = X \sin \frac{n\pi y}{b} . \tag{2-34}$$

The deflection surface produced by this variation of the moment is given by Equation (2-10), where X_n in this case is X, that is;

$$W(x, y) = \frac{X}{2D\beta_n^2 \sinh \beta_n a} \left\{ \beta_n x \cosh \beta_n (a - x) - \right.$$

$$\left. - \beta_n a \frac{\sinh \beta_n x}{\sinh \beta_n a} \right\} \sin \beta_n y. \tag{2-35}$$

Next consider an identical plate to be loaded by a concentrated load, P_c, at the point $x = \lambda a$, $y = \gamma b$. The slope variation developed at the edge $x = 0$ will be a function of y and may be expressed as a trigonometric series of the form

$$\left(\frac{\partial W_o}{\partial x}\right)_{x=0} = \sum_{m=1}^{\infty} t_{xm} \sin \frac{m\pi y}{b} , \tag{2-36}$$

where t_{xm} will be the mth-component angular load function for a concentrated load.

From Maxwell's reciprocal theorem, the work done by one system of loads on the deformations produced by another system of loads is equal to the work done by the second system on the deformations produced by the first. It is therefore concluded that

$$\int_0^b \left(M_{x=0}\right) \left(\frac{\partial W_o}{\partial x}\right)_{x=0} dy = P_c \, W(\lambda a, \gamma b) . \tag{2-37}$$

Expansion of the left side of Equation(2-37)yields

$$\int_0^b \left(M_{x=0}\right)\left(\frac{\partial W_o}{\partial x}\right)_{x=0} dy = \int_0^b X \sin\frac{n\pi y}{b} \sum_{m=1}^{\infty} t_{xm} \sin\frac{m\pi y}{b}\, dy$$

$$= X\int_0^b \Big(t_{x1} \sin\frac{n\pi y}{b} \sin\frac{\pi y}{b} + t_{x2} \sin\frac{n\pi y}{b} \sin\frac{2\pi y}{b} +$$

$$+ \cdots + t_{xn} \sin^2\frac{n\pi y}{b} + \cdots \Big)\, dy.$$

Making use of the identity

$$\int_0^b \sin\frac{n\pi y}{b} \sin\frac{m\pi y}{b}\, dy = \begin{cases} b/2 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2-38}$$

The left side of Equation(2-37) becomes

$$\int_0^b \left(M_{x=0}\right)\left(\frac{\partial W_o}{\partial x}\right)_{x=0} dy = X\, t_{xn} \frac{b}{2} . \tag{2-39}$$

Then

$$X\, t_{xn} \frac{b}{2} = P_c \left\{ \frac{X}{2D\beta_n^2 \sinh\beta_n a} \left[\beta_n \lambda a \cosh\beta_n (a - \lambda a) - \right.\right.$$

$$\left.\left. - \beta_n a \frac{\sinh\beta_n \lambda a}{\sinh\beta_n a}\right] \sin\beta_n \gamma b \right\} .$$

Simplification of this expression yields

$$t_{xn} = \frac{aP_c \sinh\beta_n \lambda a}{\pi D n \sinh\beta_n a} (\coth\beta_n a - \lambda \coth\beta_n \lambda a) \sin n\pi\gamma . \tag{2-40}$$

Equation (2-40) expresses the nth-component angular load function in the x-direction due to a concentrated load.

For a general variation of load, expressed by Equation (2-17), the deflection surface may be assumed to take on a general shape expressed by the equation

$$W_o = \sum_{n=1}^{\infty} \phi_n(x) \sin \beta_n y , \tag{2-41}$$

where $\phi_n(x)$ is a function of x. The slope at the edge x = 0 is then

$$\left(\frac{\partial W_o}{\partial x}\right)_{x=o} = \sum_{n=1}^{\infty} \left(\frac{\partial \phi_n(x)}{\partial x}\right)_{x=o} \sin \beta_n y . \tag{2-42}$$

Denoting

$$t_{xn} = \left(\frac{\partial \phi_n(x)}{\partial x}\right)_{x=o} \tag{2-43}$$

the slope becomes

$$\left(\frac{\partial W_o}{\partial x}\right)_{x=o} = \sum_{n=1}^{\infty} t_{xn} \sin \beta_n y . \tag{2-44}$$

The nth-component angular load function, t_{xn}, is interpreted physically as the maximum slope per unit length at the edge of a simply-supported plate due to the nth-component of the load distribution.

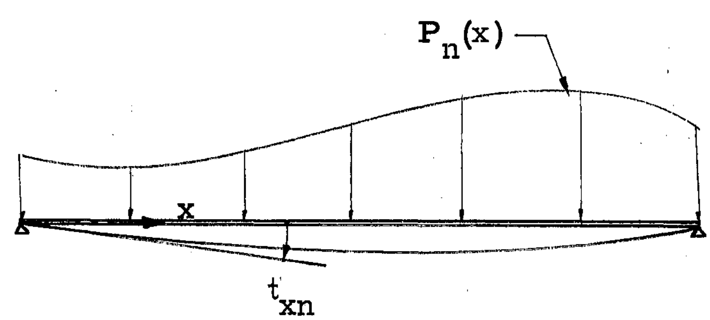


Figure 2-5

The nth-Component Angular Functions Due to Loads

CHAPTER III

THREE-MOMENT EQUATION

3-1. Derivation on the Three-Moment Equation

A continuous rectangular plate of constant thickness, subjected to a general system of transverse loads is considered. The plate is simply supported along the edges $y = 0$ and $y = b$. The supports along the lines where $x = \text{constant}$ are denoted by $0, 1, 2, \cdots, i, j, k, \cdots,$ and are assumed neither to yield to the pressure in the transverse direction nor offer any resistance to the rotation of the plate with respect to these lines. The bending moments along the intermediate supports $M_1(y), M_2(y), \cdots, M_i(y), M_j(y), M_k(y), \cdots,$ are selected as unknowns. A typical portion of this continuous plate is shown in Fig. (3-1).

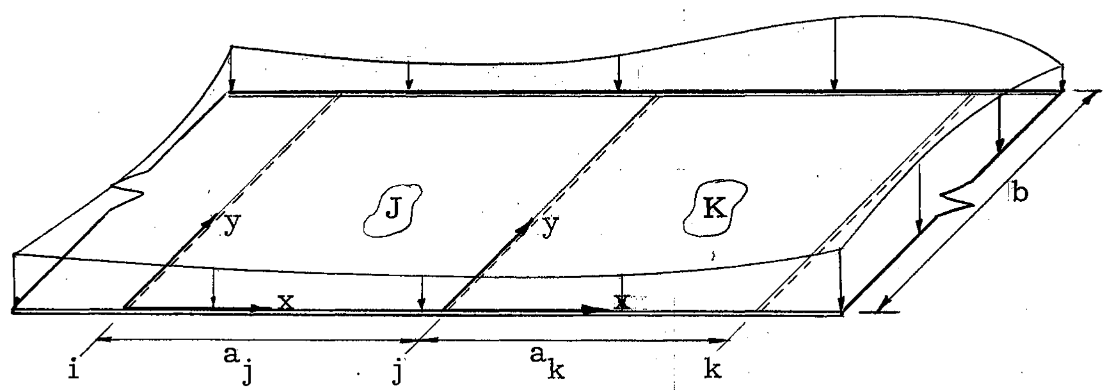


Figure 3-1

Isolated Portion of a Continuous Plate

The equilibrium state of span J (any span) may be determined by combining the solutions for the simply-supported span J due to lateral loads and due to moments distributed along the edges (Fig. 3-2).

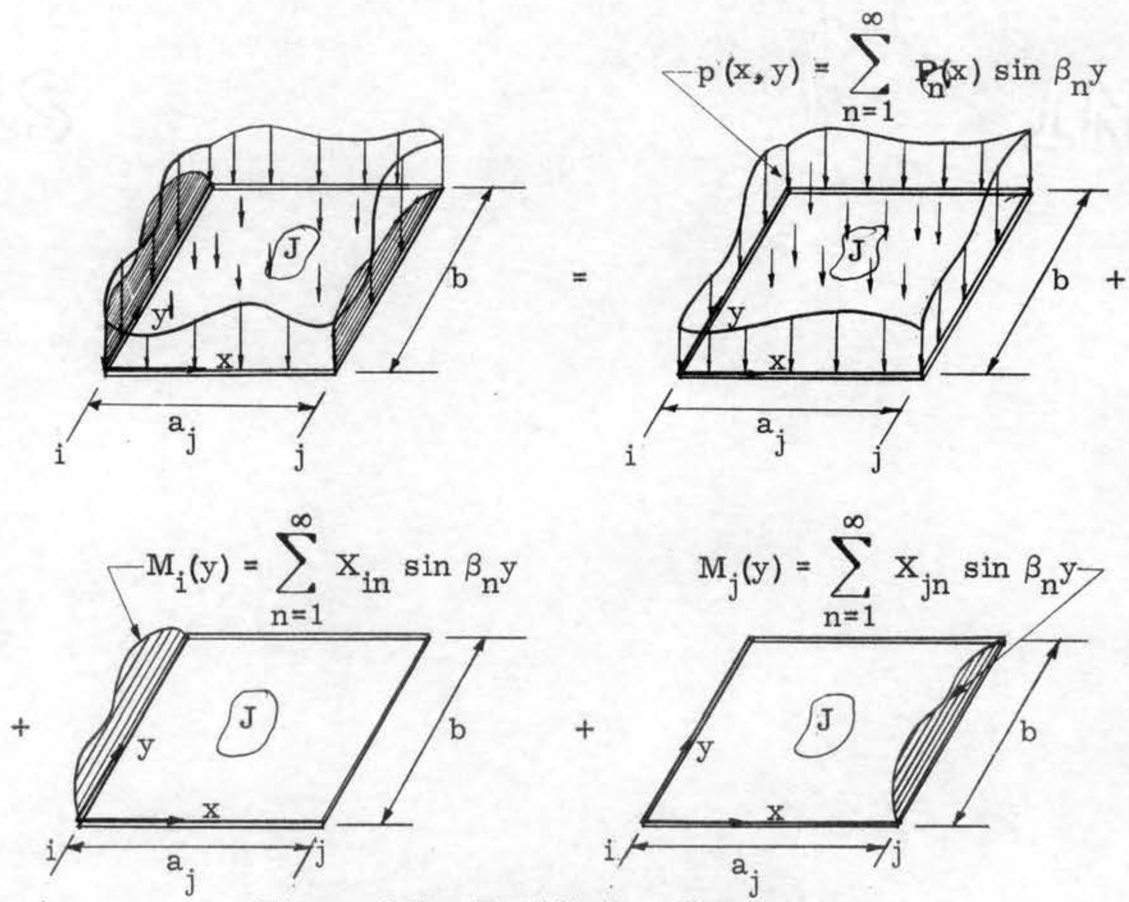


Figure 3-2. Equilibrium Components

Referring to Equations (2-10) and (2-41), the deflection surface W_J of span J may be written

$$W_J = \sum_{n=1}^{\infty} \phi_n(x) \sin \beta_n y +$$

$$+ \sum_{n=1}^{\infty} \frac{X_{in}}{2D\,\beta_n^2 \sinh \beta_n a_j} \left\{ \beta_n x \cosh \beta_n (a_j - x) - \right.$$

$$\left. - \beta_n a_j \frac{\sinh \beta_n x}{\sinh \beta_n a_j} \right\} \sin \beta_n y +$$

$$+ \sum_{n=1}^{\infty} \frac{X_{jn}}{2D\,\beta_n^2 \sinh \beta_n a_j} \Big\{ \beta_n (a_j - x) \cosh \beta_n x \; -$$

$$- \; \beta_n a_j \frac{\sinh \beta_n (a_j - x)}{\sinh \beta_n a_j} \Big\} \sin \beta_n y. \qquad (3\text{-}1)$$

The first partial derivative of Equation (3-1) with respect to x is

$$\frac{\partial W_J}{\partial x} = \sum_{n=1}^{\infty} \frac{\partial \phi_n(x)}{\partial x} \sin \beta_n y \; +$$

$$+ \sum_{n=1}^{\infty} \frac{X_{in}}{2D\,\beta_n \sinh \beta_n a_j} \Big\{ \cosh \beta_n (a_j - x) \; -$$

$$- \; \beta_n x \sinh \beta_n (a_j - x) \; - \; \beta_n a_j \frac{\cosh \beta_n x}{\sinh \beta_n a_j} \Big\} \sin \beta_n y \; +$$

$$+ \sum_{n=1}^{\infty} \frac{X_{jn}}{2D\,\beta_n \sinh \beta_n a_j} \Big\{ \cosh \beta_n x \; - \; \beta_n (a_j - x) \sinh \beta_n x \; -$$

$$- \; \beta_n a_j \frac{\cosh \beta_n (a_j - x)}{\sinh \beta_n a_j} \Big\} \sin \beta_n y. \qquad (3\text{-}2)$$

The slopes at the edges i and j are found by evaluating Equation (3-2) at $x = 0$ and $x = a_j$, respectively.

$$\left(\frac{\partial W_J}{\partial x} \right)_{x=o} = \sum_{n=1}^{\infty} \phi_{ijn} \sin \beta_n y$$

$$(3\text{-}3)$$

$$- \left(\frac{\partial W_J}{\partial x} \right)_{x=a_j} = \sum_{n=1}^{\infty} \phi_{jin} \sin \beta_n y \; ,$$

where, referring to Equations (2-15), (2-16), and (2-44),

$$\phi_{ijn} = t_{ijn} + X_{in} f_{ijn} + X_{jn} g_{ijn}$$

$$\phi_{jin} = t_{jin} + X_{jn} f_{jin} + X_{in} g_{jin} \tag{3-4}$$

Likewise, for span K the slopes at j and k are

$$\left(\frac{\partial W_K}{\partial x}\right)_{x=o} = \sum_{n=1}^{\infty} \phi_{jkn} \sin \beta_n y$$

$$-\left(\frac{\partial W_K}{\partial x}\right)_{x=a_k} = \sum_{n=1}^{\infty} \phi_{kjn} \sin \beta_n y \tag{3-5}$$

where

$$\phi_{jkn} = t_{jkn} + X_{jn} f_{jkn} + X_{kn} g_{jkn}$$

$$\phi_{kjn} = t_{kjn} + X_{kn} f_{kjn} + X_{jn} g_{kjn} \tag{3-6}$$

The equilibrium states of spans J and K (any two adjacent spans) are now combined such that geometrical compatibility is satisfied at the intersection of the two spans. This implies that

$$(W_J)_{x=a_j} = (W_K)_{x=o} \tag{3-7a}$$

$$\left(\frac{\partial W_J}{\partial x}\right)_{x=a_j} = \left(\frac{\partial W_K}{\partial x}\right)_{x=o} \tag{3-7b}$$

$$\left(\frac{\partial^2 W_J}{\partial x^2}\right)_{x=a_j} = \left(\frac{\partial^2 W_K}{\partial x^2}\right)_{x=o} \tag{3-7c}$$

$$\left(\frac{\partial W_J}{\partial y}\right)_{x=a_j} = \left(\frac{\partial W_K}{\partial y}\right)_{x=o} \tag{3-7d}$$

$$\left(\frac{\partial^2 W_J}{\partial y^2}\right)_{x=a_j} = \left(\frac{\partial^2 W_K}{\partial y^2}\right)_{x=o} \tag{3-7e}$$

$$\left(\frac{\partial^2 W_J}{\partial x \partial y}\right)_{x=a_j} = \left(\frac{\partial^2 W_K}{\partial x \partial y}\right)_{x=o} . \tag{3-7f}$$

Equations (3-7a), (3-7d), and (3-7e) are automatically satisfied in that the equilibrium states of the two individual spans were found for the simply supported case. Equation (3-7c) is satisfied by the assumption made earlier that the support j offer no resistance to rotation, and (3-7f) will be satisfied if (3-7a) and 3-7b) hold true. The continuity at support j depends, therefore, on the equivalence

$$\left(\frac{\partial W_J}{\partial x}\right)_{x=a_j} = \left(\frac{\partial W_K}{\partial x}\right)_{x=o}$$

or

$$- \sum_{n=1}^{\infty} \phi_{jin} \sin \beta_n y = \sum_{n=1}^{\infty} \phi_{jkn} \sin \beta_n y .$$

Equating the coefficients of like harmonics yields

$$- \phi_{jin} = \phi_{jkn} , \quad n = 1, 2, 3, \cdots , \tag{3-8}$$

from which,

$$- t_{jin} - X_{jn} f_{jin} - X_{in} g_{ijn} = t_{jkn} + X_{jn} f_{jkn} + X_{kn} g_{jkn} .$$

or

$$X_{in} g_{jin} + X_{jn}(f_{jin} + f_{jkn}) + X_{kn} g_{jkn} = - (t_{jin} + t_{jkn}) . \tag{3-9}$$

From the reciprocal theorem of slopes

$$\left.\begin{aligned} g_{jin} &= g_{ijn} \\ g_{jkn} &= g_{kjn} \,. \end{aligned}\right\} \qquad (3\text{-}10)$$

Equation (3-9) may now be rewritten as

$$X_{in}\, g_{ijn} + X_{jn} \sum (f_j)_n + X_{kn}\, g_{kjn} = -\sum (t_j)_n \,, \qquad (3\text{-}11)$$

where

$$\left.\begin{aligned} \sum (f_j)_n &= f_{jin} + f_{jkn} \\ \sum (t_j)_n &= t_{jin} + t_{jkn} \,. \end{aligned}\right\} \qquad (3\text{-}12)$$

Equation (3-11) is the general three-moment equation for the nth-mode of moments due to the nth component of loading on the plate.

3-2. Carry-over Moment Equation

Solving Equation (3-11) for X_{jn} yields

$$X_{jn} = -\frac{g_{ijn}}{\sum (f_j)_n} X_{in} + \frac{\sum (t_j)_n}{\sum (f_j)_n} - \frac{g_{kjn}}{\sum (f_j)_n} X_{kn} \,. \qquad (3\text{-}13)$$

Defining

$$\left.\begin{aligned} r_{ijn} &= -\frac{g_{ijn}}{\sum (f_j)_n} \\ r_{kjn} &= -\frac{g_{jkn}}{\sum (f_j)_n} \\ x^*_{jn} &= -\frac{\sum (t_j)_n}{\sum (f_j)_n} \,, \end{aligned}\right\} \qquad (3\text{-}14)$$

Equation (3-13) becomes

$$X_{jn} = r_{ijn} X_{in} + x^*_{jn} + r_{kjn} X_{kn} . \tag{3-15}$$

This equation is the carry-over form of the general three-moment equation for the nth mode of moments.

The final redundant moment along the support j is

$$M_j(y) = \sum_{n=1}^{\infty} X_{jn} \sin \frac{n\pi y}{b} \tag{3-16}$$

3-3. Discussion and Procedure

Reviewing the development of Equation (3-15), it is seen that the mechanics of its formulation involve physical concepts which give meaning to the equation. For a given component of loading:

1.) The continuous plate is assumed to consist of a series of simply-supported plates.
2.) To establish continuity of slopes along the support j, with supports i and k remaining hinged, a moment of amplitude x^*_{jn} is applied at the support j.
3.) When continuity of slopes is established at supports i and k, the moments at those supports will be carried over to support j. This procedure is continued until full continuity of the elastic surface of the plate is established.

The moment amplitude x^*_{jn} is termed the starting moment for the nth component. The quantities r_{ijn} and r_{kjn} are called the carry-over factors from supports i to j and k to j, respectively, for the nth component.

The carry-over moment equations, for a given number of redundant moments and a specified term of the series, constitute a system of equations in iteration form. The procedure for the solution of the system is a numerical, successive approximation based on the physical model described above and may be carried out to a desired degree of accuracy.

For each term in the infinite trigonometric series for the plate, there exists a solution set of amplitudes for the redundant moment components. Each solution set may be obtained from a carry-over procedure.

A typical carry-over pattern is shown in Figure 3-3 for an nth term. This pattern is the same as that used in the analysis of continuous beams by the Carry-Over Moment Method (2).

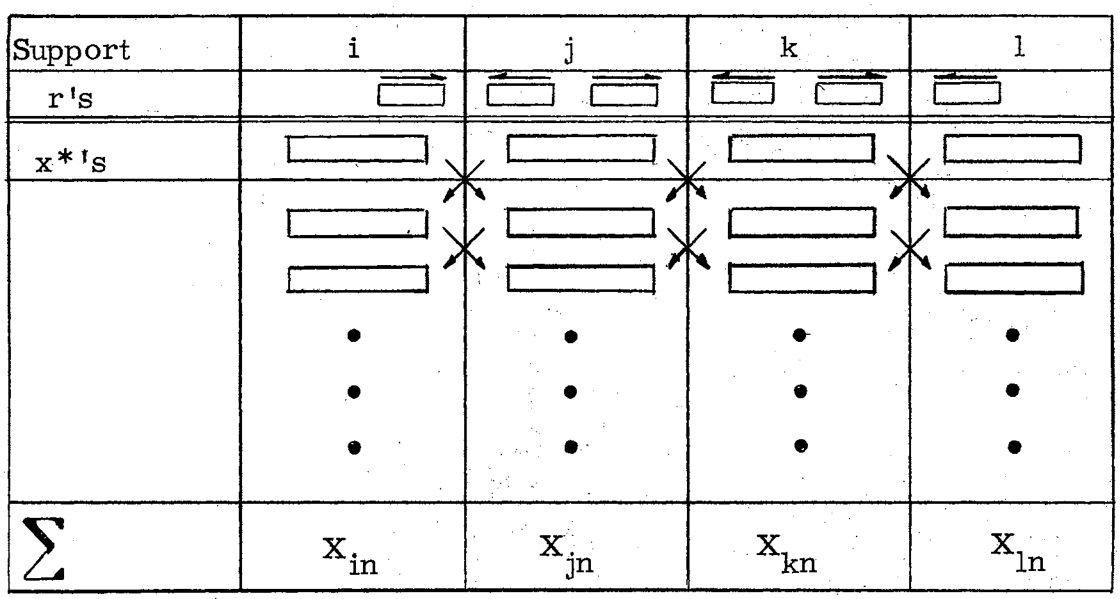


Figure 3-3

Typical Carry-Over Pattern

As the number of terms of the series increases, or for large length-width ratios, the angular carry-over values become small in comparison to the angular flexibilities, and the carry-over factors approach zero. In which case the Carry-Over Moment Equation becomes

$$X_{jn} = x^{*}_{jn} . \tag{3-17}$$

CHAPTER IV

PLATE CONSTANTS

4-1. General Notes

The constants entering into the analysis by means of the carry-over moment procedure were programmed for the IBM 650 Digital Computer. The numerical values of these constants for a common range of length-width ratios appear in the tables which comprise this chapter.

4-2. Tables 1, 2, ···, 12, 13

Plate constants for the first nine intergers of n are recorded in Tables 1, 2, ···, 12, 13. The geometry of the plate is given in a sketch at the top of each table. The dimensions are:

a = length of the plate,

b = width of the plate,

h = thickness.

The flexural rigidity is

$$D = \frac{Eh^3}{12(1 - \mu^2)}, \quad \text{where} \quad E = \text{modulas of elasticity}$$

$$\mu = \text{poisson's ratio.}$$

The plate constant formulas listed in each table are:

a.) The nth-component Angular Flexibilities (Eq. 2-14)

$$f_{ijn} = f_{jin} = f_{xn} = C_1 \frac{b}{D}$$

C_1 = angular flexibility coefficient.

b.) The nth-Component Angular Carry-Over Values (Eq. 2-14)

$$g_{ijn} = g_{jin} = g_{xn} = C_2 \frac{b}{D}$$

C_2 = angular carry-over value coefficient.

c.) The nth-Component Angular Load Functions (Eq. 2-43)

1.) Due to a Concentrated Load (Eq. 2-40)

$$t_{ijn}^{(LL)} = C_4 \frac{Pa}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

C_4 = left end slope coefficient due to a concentrated load of magnitude P at the point yx.

C_5 = right end slope coefficient due to a concentrated load of magnitude P at the point yx.

2.) Due to a Uniformly Distributed Load (Eq. 2-32)

$$t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

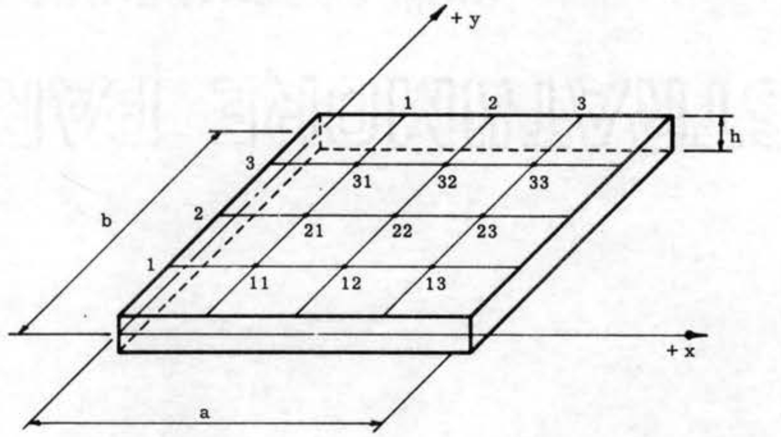
C_3 = end slope coefficient due to a uniformly distributed load of intensity w.

When the coefficients are identically zero, a dash is recorded in the entry position.

4-3. Plate Constant Tables

The numerical coefficients for the angular functions used in the analysis by the Carry-Over Moment Method are presented in the following tables:

TABLE I $\frac{a}{b} = 1.0$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

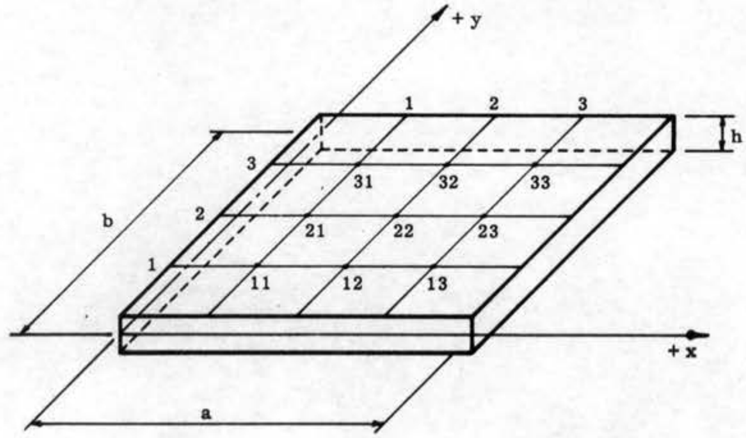
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C_4											
n \ yx	11	12	13	21	22	23	31	32	33	C_1	C_2	C_3
1	+.02447	+.02057	+.01054	+.03462	+.02909	+.01490	+.02447	+.02057	+.01054	+.15600	+.02968	+.00685
2	-.00827	+.00307	+.00100	-	-	-	-.00827	+.00307	+.00100	+.07957	+.00157	-
3	+.00178	+.00033	+.00005	-.00254	-.00048	-.00007	+.00178	+.00033	+.00005	+.05305	+.00007	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00022	.00000	.00000	+.00031	+.00001	.00000	-.00022	.00000	.00000	+.03183	.00000	+.00002
6	+.00012	.00000	.00000	-	-	-	+.00012	.00000	.00000	+.02653	.00000	-
7	-.00003	.00000	.00000	-.00004	.00000	.00000	-.00003	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	+.00001	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE 2 $\frac{a}{b} = 1.1$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

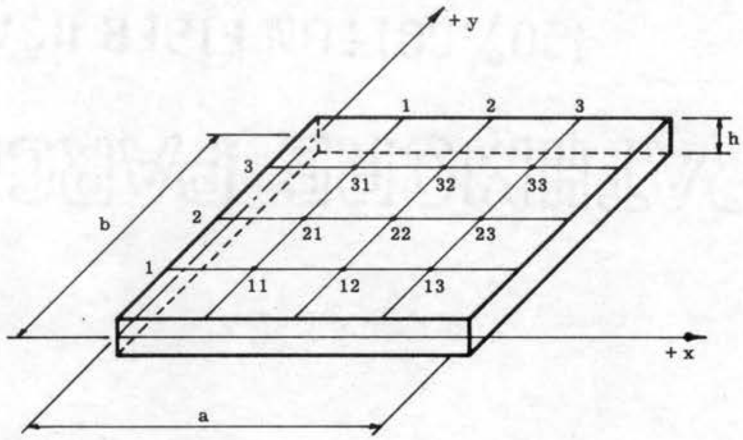
$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C_4									C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.02300	+.01987	+.00893	+.03310	+.02573	+.01231	+.02300	+.01987	+.00893	+.15728	+.02477	+.00753
2	-.00707	+.00201	+.00063	-	-	-	-.00707	+.00201	+.00063	+.07958	+.00094	-
3	+.00140	+.00021	+.00002	-.00200	-.00044	-.00004	+.00140	+.00021	+.00002	+.05305	+.00003	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00015	.00000	.00000	+.00021	.00000	.00000	-.00015	.00000	.00000	+.03183	.00000	+.00002
6	+.00007	.00000	.00000	-	-	-	+.00007	.00000	.00000	+.02653	.00000	-
7	-.00002	.00000	.00000	-.00003	.00000	.00000	-.00002	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	+.00001	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE 3

$\frac{a}{b} = 1.2$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

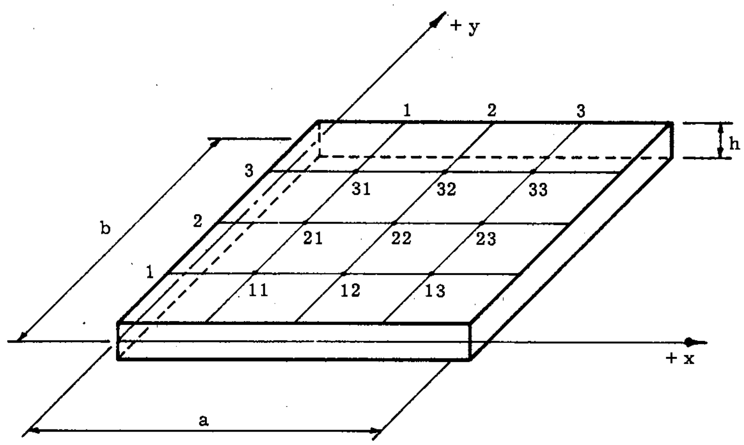
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C_4									C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.02149	+.01594	+.00748	+.03122	+.02256	+.01054	+.02149	+.01594	+.00748	+.15804	+.02037	+.00810
2	-.00604	+.00163	+.00040	-	-	-	-.00604	+.00123	+.00040	+.07958	+.00055	-
3	+.00111	+.00013	+.00001	-.00159	-.00019	-.00002	+.00111	+.00013	+.00001	+.05305	+.00001	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00010	.00000	.00000	+.00012	.00000	.00000	-.00010	.00000	.00000	+.03183	.00000	+.00002
6	+.00005	.00000	.00000	-	-	-	+.00005	.00000	.00000	+.02653	.00000	-
7	-.00001	.00000	.00000	-.00001	.00000	.00000	-.00001	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE 4 $\frac{a}{b} = 1.3$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D}$$

$$t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

$$g_{ijn} = g_{jin} = C_2 \frac{b}{D}$$

$$t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)}$$

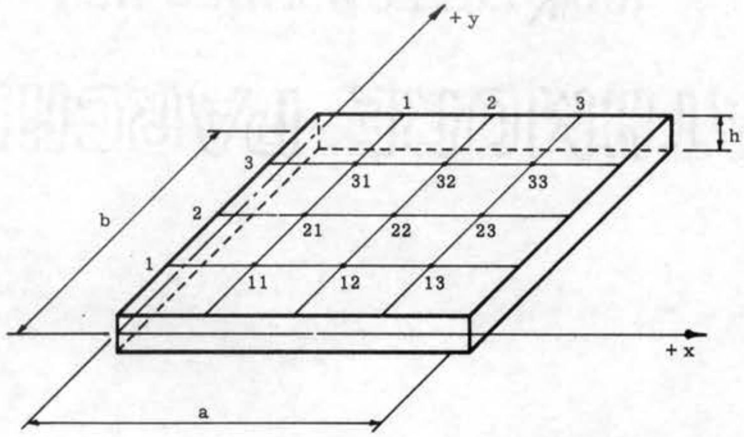
$$t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

Influence Coefficients C_4										C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.02001	+.01390	+.00619	+.02828	+.01964	+.00854	+.02001	+.01390	+.00619	+.15851	+.01655	+.00856
2	-.00516	+.00141	+.00025	-	-	-	-.00516	+.00141	+.00025	+.07958	+.00032	-
3	+.00088	+.00021	+.00001	-.00125	-.00011	-.00002	+.00088	+.00021	+.00001	+.05305	+.00001	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00007	.00000	.00000	+.00010	.00000	.00000	-.00007	.00000	.00000	+.03183	.00000	+.00002
6	+.00003	.00000	.00000	-	-	-	+.00003	.00000	.00000	+.02653	.00000	-
7	-.00001	.00000	.00000	-.00002	.00000	.00000	-.00001	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
Influence Coefficients C_5												

TABLE 5

$\frac{a}{b} = 1.4$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

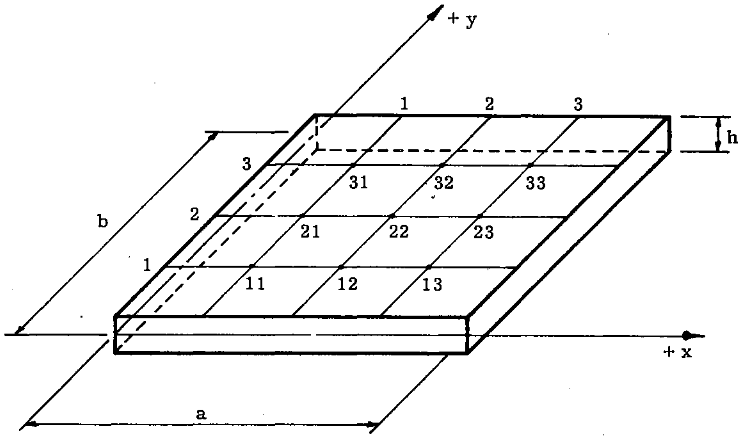
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C_4									C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.01859	+.01203	+.00509	+.02627	+.01701	+.00702	+.01859	+.01203	+.00509	+.15878	+.01331	+.00893
2	-.00441	+.00121	+.00016	-	-	-	-.00441	+.00121	+.00016	+.07958	+.00019	-
3	+.00069	+.00003	.00000	-.00099	-.00007	-.00001	+.00069	+.00003	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00005	.00000	.00000	+.00007	.00000	.00000	-.00005	.00000	.00000	+.03183	.00000	+.00002
6	+.00002	.00000	.00000	-	-	-	+.00002	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	-.00001	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE 6 $\frac{a}{b} = 1.5$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

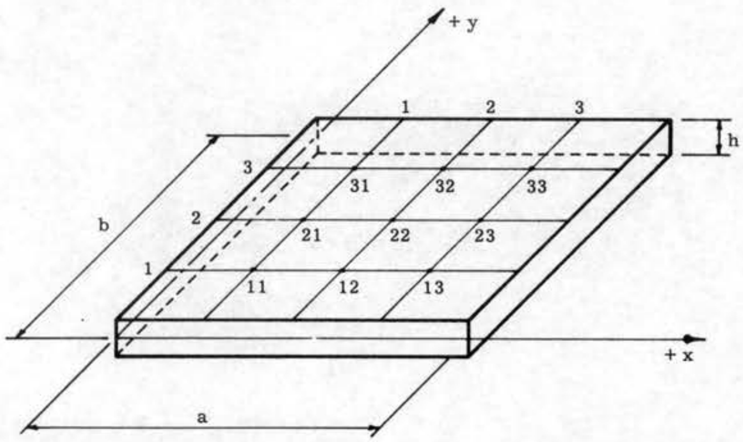
$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C_4									C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.01723	+.01038	+.00415	+.02433	+.01468	+.00572	+.01723	+.01038	+.00415	+.15894	+.01062	+.00923
2	-.00377	+.00105	+.00010	-	-	-	-.00377	+.00105	+.00010	+.07958	+.00011	-
3	+.00055	+.00002	.00000	-.00079	-.00005	-.00001	+.00055	+.00002	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00003	.00000	.00000	+.00004	.00000	.00000	-.00003	.00000	.00000	+.03183	.00000	+.00002
6	+.00001	.00000	.00000	-	-	-	+.00001	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	-.00001	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE 7

$\frac{a}{b} = 1.6$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D}$$

$$t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

$$g_{ijn} = g_{jin} = C_2 \frac{b}{D}$$

$$t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

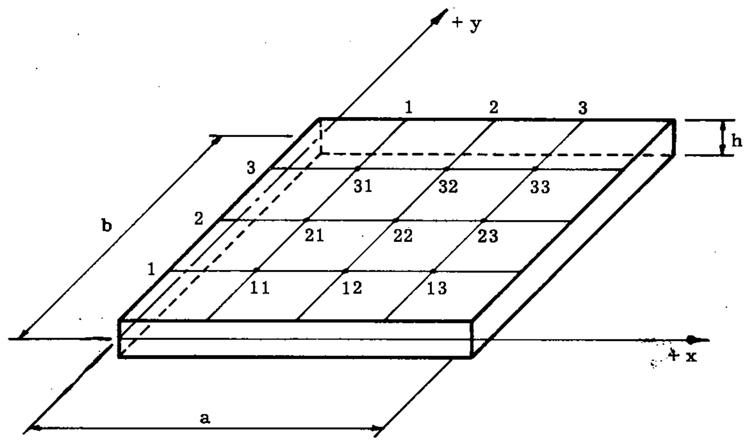
$$D = \frac{Eh^3}{12(1 - \mu^2)}$$

$$t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

Influence Coefficients C_4												
n \ yx	11	12	13	21	22	23	31	32	33	C_1	C_2	C_3
1	+.01596	+.00895	+.00337	+.02252	+.01264	+.00465	+.01596	+.00895	+.00337	+.15903	+.00841	+.00946
2	-.00322	+.00091	+.00006	-	-	-	-.00322	+.00091	+.00006	+.07958	+.00006	-
3	+.00043	+.00001	.00000	-.00061	-.00003	.00000	+.00043	+.00001	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00002	.00000	.00000	+.00003	.00000	.00000	-.00002	.00000	.00000	+.03183	.00000	+.00002
6	+.00001	.00000	.00000	-	-	-	+.00001	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	-.00001	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
Influence Coefficients C_5												

TABLE 8 $\frac{a}{b} = 1.7$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t^{(LL)}_{ijn} = C_4 \frac{Pa}{D}$$

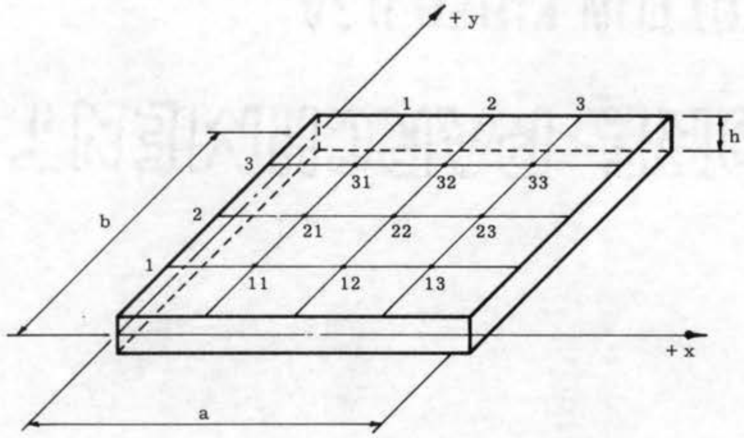
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t^{(LL)}_{jin} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)} \qquad t^{(UL)}_{ijn} = t^{(UL)}_{jin} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

Influence Coefficients C4										C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.01477	+.00768	+.00272	+.02087	+.01086	+.00375	+.01477	+.00768	+.00272	+.15908	+.00662	+.00965
2	-.00275	+.00078	+.00004	-	-	-	-.00275	+.00078	+.00004	+.07958	+.00004	-
3	+.00034	+.00001	.00000	-.00049	-.00002	.00000	+.00034	+.00001	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00001	.00000	.00000	+.00002	.00000	.00000	-.00001	.00000	.00000	+.03183	.00000	+.00002
6	.00000	.00000	.00000	-	-	-	.00000	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
Influence Coefficients C5												

TABLE 9 $\qquad \frac{a}{b} = 1.8$



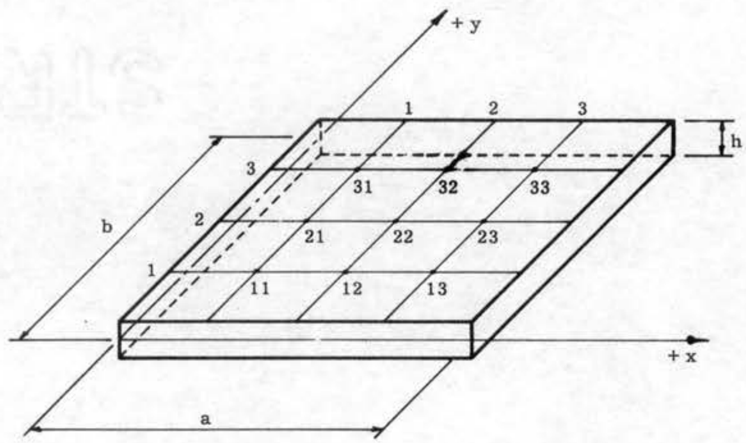
$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions												
	Influence Coefficients C_4											
n \ yx	11	12	13	21	22	23	31	32	33	C_1	C_2	C_3
1	+.01367	+.00659	+.00219	+.01933	+.00932	+.00302	+,01367	+.00659	+.00219	+.15911	+.00519	+.00979
2	-.00235	+.00067	+.00002	-	-	-	-.00235	+.00067	+.00002	+.07958	+.00002	-
3	+.00027	.00000	.00000	-.00039	-.00001	.00000	+.00027	.00000	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00001	.00000	.00000	+.00001	.00000	.00000	-.00001	.00000	.00000	+.03183	.00000	+.00002
6	.00000	.00000	.00000	-	-	-	.00000	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE 10 $\frac{a}{b} = 1.9$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

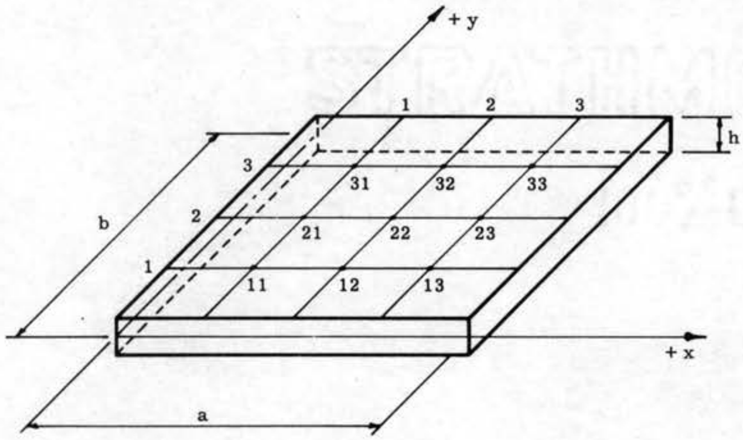
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C_4											
n \ yx	11	12	13	21	22	23	31	32	33	C_1	C_2	C_3
1	+.01264	+.00565	+.00176	+.01787	+.00799	+.00243	+.01264	+.00565	+.00176	+.15913	+.00404	+.00990
2	-.00201	+.00057	+.00002	-	-	-	-.00201	+.00057	+.00002	+.07958	+.00001	-
3	+.00021	.00000	.00000	-.00030	-.00001	.00000	+.00021	.00000	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	-.00001	.00000	.00000	+.00001	.00000	.00000	-.00001	.00000	.00000	+.03183	.00000	+.00002
6	.00000	.00000	.00000	-	-	-	.00000	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C_5											

TABLE II $\quad \frac{a}{b} = 2.0$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

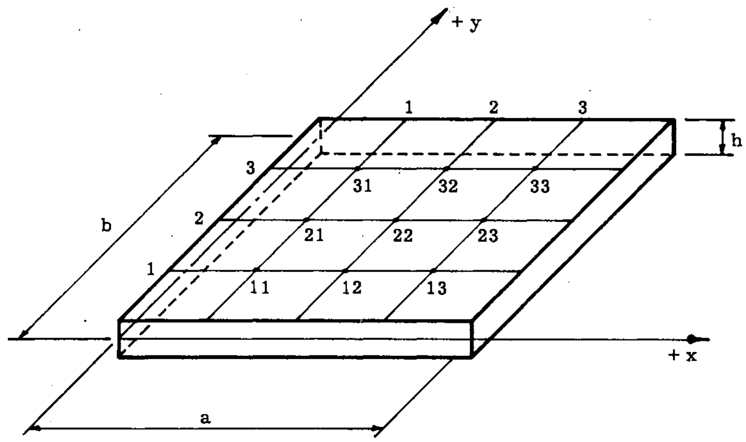
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C4											
n \ yx	11	12	13	21	22	23	31	32	33	C_1	C_2	C_3
1	+.01169	+.00484	+.00141	+.01652	+.00684	+.00194	+.01169	+.00484	+.00141	+.15914	+.00314	+.00999
2	-.00172	+.00049	+.00001	-	-	-	-.00172	+.00049	+.00001	+.07958	+.00001	-
3	+.00017	.00000	.00000	-.00024	.00000	.00000	+.00017	.00000	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	.00000	.00000	.00000	+.00001	.00000	.00000	.00000	.00000	.00000	+.03183	.00000	.00002
6	.00000	.00000	.00000	-	-	-	.00000	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C5											

TABLE 12 $\qquad \frac{a}{b} = 3.0$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

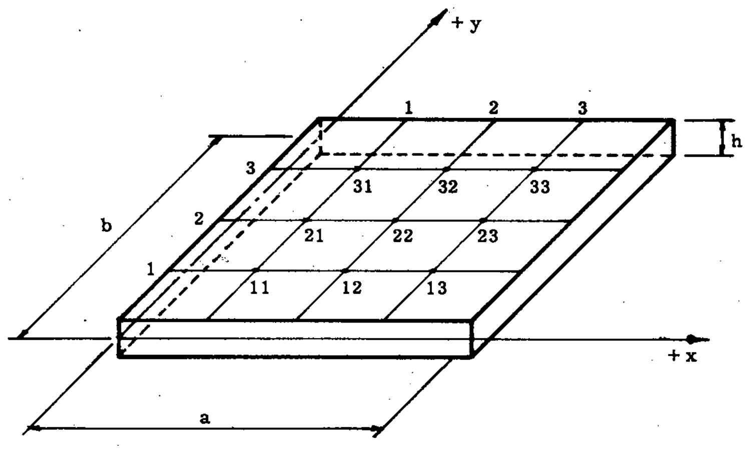
$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

	Influence Coefficients C4									C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.00533	+.00101	+.00014	+.00752	+.00143	+.00019	+.00533	+.00101	+.00014	+.15915	+.00022	+.01025
2	-.00036	+.00010	.00000	-	-	-	-.00036	+.00010	.00000	+.07958	.00000	-
3	+.00002	.00000	.00000	-.00003	.00000	.00000	+.00002	.00000	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.03183	.00000	+.00002
6	.00000	.00000	.00000	-	-	-	.00000	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	-	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
	Influence Coefficients C5											

TABLE 13 $\qquad \frac{a}{b} = 4.0$



$$f_{ijn} = f_{jin} = C_1 \frac{b}{D} \qquad t_{ijn}^{(LL)} = C_4 \frac{Pa}{D}$$

$$g_{ijn} = g_{jin} = C_2 \frac{b}{D} \qquad t_{jin}^{(LL)} = C_5 \frac{Pa}{D}$$

$$D = \frac{Eh^3}{12(1 - \mu^2)} \qquad t_{ijn}^{(UL)} = t_{jin}^{(UL)} = C_3 \frac{2wb^3}{D}$$

Coefficients of Angular Functions

Influence Coefficients C_4										C_1	C_2	C_3
n \ yx	11	12	13	21	22	23	31	32	33			
1	+.00243	+.00021	+.00001	+.00343	+.00030	+.00001	+.00243	+.00021	+.00001	+.15915	+.00001	+.01027
2	-.00007	+.00002	.00000	-	-	-	-.00007	+.00002	.00000	+.07958	.00000	-
3	.00000	.00000	.00000	-.00001	.00000	.00000	.00000	.00000	.00000	+.05305	.00000	+.00013
4	-	-	-	-	-	-	-	-	-	+.03979	.00000	-
5	.00000	.00000	+.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.03183	.00000	+.00002
6	.00000	.00000	.00000	-	-	-	.00000	.00000	.00000	+.02653	.00000	-
7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.02274	.00000	.00000
8	[illegible]	-	-	-	-	-	-	-	-	+.01989	.00000	-
9	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	+.01768	.00000	.00000
n \ yx	13	12	11	23	22	21	33	32	31	C_1	C_2	C_3
Influence Coefficients C_5												

CHAPTER V

NUMERICAL EXAMPLE

5-1. General Notes

A numerical example is presented to illustrate the procedure of analysis by the Carry-Over Moment Method. References are made to the equations and tables that are used. Units for the various values are in terms of pounds, feet, or pound-feet per foot.

Example: A four span continuous plate of constant rigidity, resting on rigid simple supports loaded as shown (Fig. 5-1) is analyzed. The load distribution on each span is symmetrical with respect to the line $y = \frac{b}{2} = 5$ feet. Therefore the even terms of the series solutions for the redundant moments vanish.

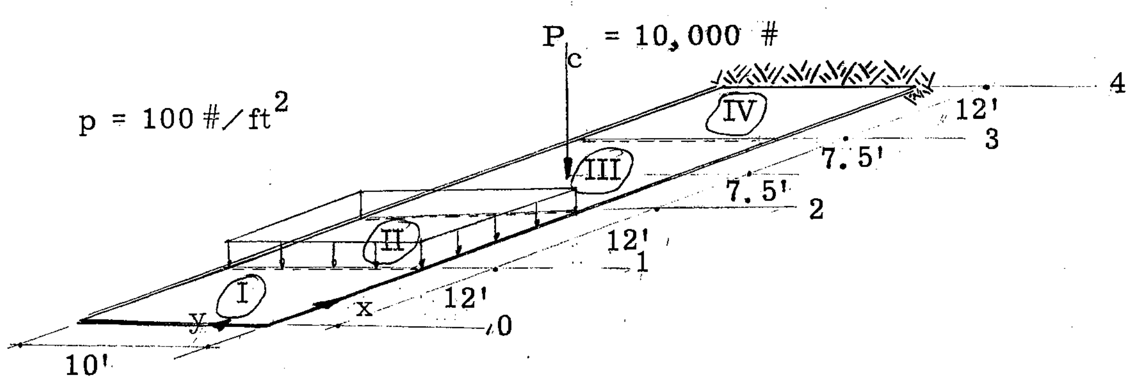


Figure 5-1

Four Span Continuous Plate

5-2. Angular Moment Functions (Eqs. 2-14)(Tables 4-3, 4-6)

$$f_{01n} = f_{10n} = f_{12n} = f_{21n} = f_{34n} = f_{43n}$$

$$= \begin{cases} (.15804)\ \frac{10}{D} & \text{for } n = 1 \\ (.05305)\ \frac{10}{D} & \text{for } n = 3 \\ (.03183)\ \frac{10}{D} & \text{for } n = 5 \end{cases}$$

$$f_{23n} = f_{32n} = \begin{cases} (.15894)\ \frac{10}{D} & \text{for } n = 1 \\ (.05305)\ \frac{10}{D} & \text{for } n = 3 \\ (.03183)\ \frac{10}{D} & \text{for } n = 5 \end{cases}$$

$$g_{01n} = g_{10n} = g_{12n} = g_{21n} = g_{34n} = g_{43n}$$

$$= \begin{cases} (.02037)\ \frac{10}{D} & \text{for } n = 1 \\ (.00001)\ \frac{10}{D} & \text{for } n = 3 \\ (.00000)\ \frac{10}{D} & \text{for } n = 5 \end{cases}$$

$$g_{23n} = g_{32n} = \begin{cases} (.01062)\ \frac{10}{D} & \text{for } n = 1 \\ (.00000)\ \frac{10}{D} & \text{for } n = 3 \\ (.00000)\ \frac{10}{D} & \text{for } n = 5 \end{cases}$$

5-3. Angular Load Functions (Eqs. 2-32, 2-40)(Tables 4-3, 4-6)

$$t_{01n} = t_{10n} = t_{34n} = t_{43n} = 0$$

$$t_{12n} = t_{21n} = \begin{cases} (162.00)\ \frac{10}{D} & \text{for } n = 1 \\ (2.60)\ \frac{10}{D} & \text{for } n = 3 \\ (.40)\ \frac{10}{D} & \text{for } n = 5 \end{cases}$$

$$t_{32n} = t_{23n} = \begin{cases} (220.20)\ \frac{10}{D} & \text{for } n = 1 \\ (-.75)\ \frac{10}{D} & \text{for } n = 3 \\ 0\ \frac{10}{D} & \text{for } n = 5 \end{cases}$$

5-4. Carry-Over Factors (Eqs. 3-14)

r_{01n} = inexistant (modified)

$r_{10n} = 0$

$$r_{12n} = r_{43n} = \begin{cases} -.0001 & \text{for } n = 1 \\ -.0643 & \text{for } n = 3 \\ .0000 & \text{for } n = 5 \end{cases}$$

$$r_{21n} = \begin{cases} -.0644 & \text{for } n = 1 \\ -.0001 & \text{for } n = 3 \\ .0000 & \text{for } n = 5 \end{cases}$$

$$r_{23n} = r_{32n} = \begin{cases} -.0335 & \text{for } n = 1 \\ .0000 & \text{for } n = 3 \\ .0000 & \text{for } n = 5 \end{cases}$$

$$r_{34n} = \begin{cases} -.1289 & \text{for } n = 1 \\ -.0002 & \text{for } n = 3 \\ .0000 & \text{for } n = 5 \end{cases}$$

5-5. Starting Moments (Eqs. 3-14)

$$x^*_{0n} = 0 \qquad\qquad x^*_{4n} = 0$$

$$x^*_{1n} = \begin{cases} -\ 512.53 & \text{for } n = 1 \\ -\ \ 24.51 & \text{for } n = 3 \\ -\ \ \ 6.28 & \text{for } n = 5 \end{cases}$$

$$x^*_{2n} = \begin{cases} -\ 1205.75 & \text{for } n = 1 \\ -\ \ 17.44 & \text{for } n = 3 \\ -\ \ \ 6.28 & \text{for } n = 5 \end{cases}$$

$$x^*_{3n} = \begin{cases} -\ 694.68 & \text{for } n = 1 \\ +\ \ \ 7.07 & \text{for } n = 3 \\ \ \ \ \ 0 & \text{for } n = 5 \end{cases}$$

5-6. Carry-Over Moment Procedure (Eq. 3-15)(Fig. 3-3)

a.) For n = 1

Support	1	2	3	4
r's	-.0643	-.0644 -.0335	-.0335 -.1289	-.0643
x^*'s	- 512.53	- 1205.75	- 694.68	
	+ 77.65		+ 40.39	
	- 434.88		- 654.29	
		+ 21.92		+ 84.34
		+ 27.96		
		+ 49.88	- 5.42	
	- 3.21		- 1.77	
		+ .21	- 7.19	
		+ .24		+ .93
		+ .45	- .06	
	- .03		- .01	
			- .07	
				+ .01
$\sum$	- 438.12	- 1155.42	- 661.55	+ 85.28

Thus,

$$X_{11} = -\ 438.12 \qquad\qquad X_{21} = -\ 1155.42$$

$$X_{31} = -\ 661.55 \qquad\qquad X_{41} = +\ \ 85.28 \ .$$

b.) For n = 3

The carry-over factors for $n = 3$ have converged to a negligible value in comparison to those for $n = 1$. Therefore the amplitudes of the moments for this and succeeding components become the respective starting moment amplitudes for that component. Then

$$X_{13} = X^*_{13} = -24.51 \qquad X_{23} = X^*_{23} = -17.44$$

$$X_{33} = X^*_{33} = +7.07 \qquad X_{43} = X^*_{43} = 0 .$$

c.) For n = 5

$$X_{15} = X^*_{15} = -6.28 \qquad X_{25} = X^*_{25} = -6.28$$

$$X_{35} = X^*_{35} = 0 \qquad X_{45} = X^*_{45} = 0 .$$

6-7. Numerical Control

The values for the moment amplitudes obtained from the carry-over moment procedure must satisfy Equation (3-15).

$$X_{11} = -438.12 = (-512.53) + (-.0644)(-1155.42)$$

$$X_{21} = -1155.42 = (-.0643)(-438.12) + (-1205.75) + \\ + (-.0335)(-661.55)$$

$$X_{31} = -661.55 = (-.0335)(-1155.42) + (-694.68) + \\ + (-.0643)(85.28)$$

$$X_{41} = 85.28 = (-.1289)(-661.55) .$$

6-8. Final Moments

The final redundant moments along the supports 1, 2, 3, and 4 for three term approximations are

$$M_1(y) = -438.12 \sin\frac{\pi y}{b} - 24.51 \sin\frac{3\pi y}{b} - 6.28 \sin\frac{5\pi y}{b}$$

$$M_2(y) = -1155.42 \sin\frac{\pi y}{b} - 17.44 \sin\frac{3\pi y}{b} - 6.28 \sin\frac{5\pi y}{b}$$

$$M_3(y) = -661.55 \sin\frac{\pi y}{b} + 7.07 \sin\frac{3\pi y}{b}$$

$$M_4(y) = 85.28 \sin\frac{\pi y}{b} .$$

5-9. Discussion

The deflection at the midpoints of each span may be determined by evaluation of Equation (3-1) at $x = \frac{a}{2}$ and $y = \frac{b}{2}$.

The moments in the x and y directions at the midpoints of each span may also be found by evaluation of the following expressions at $x = \frac{a}{2}$ and $y = \frac{b}{2}$.

$$M_x = -D\left(\frac{\partial^2 W}{\partial x^2} + \mu \frac{\partial^2 W}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 W}{\partial y^2} + \mu \frac{\partial^2 W}{\partial x^2}\right) .$$

CHAPTER VI

SUMMARY AND CONCLUSIONS

The analysis of rectangular plates continuous in one direction over rigid simple supports by the Carry-Over Moment Method is presented in this thesis. General aspects of the study may be summarized as follows:

1. The angular functions for rectangular plates were developed and were used to describe the equilibrium state of a simple plate acted upon by lateral loads and bending moments distributed along its edges. The equilibrium states of two adjacent spans of a continuous plate were combined such that geometrical compatibility at the intersection of the spans was satisfied. This yielded the general three-moment equation for a continuous plate.

2. Tabulated coefficients of plate constants for a common range of length-width ratios were included to shorten the task of computing the functions used in the analysis.

3. The procedure relates the analysis of plates continuous in one direction to the analysis of continuous beams which was developed by Tuma (2).

4. The number of carry-over procedures required for a given precision is limited to a relative few because of the rapid convergency of the series. Also, as the number of terms in the series increases the carry-over moment equation reduces to an explicit solution for the respective moment components.

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